Imperial College
London

Lecture 5

Sampling

Peter Cheung
Dyson School of Design Engineering
Imperial College London

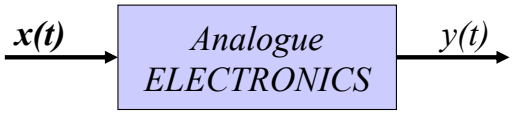
URL: www.ee.ic.ac.uk/pcheung/teaching/DE2_EE/
E-mail: p.cheung@imperial.ac.uk

In this lecture, we will examine two important topics in signal processing:

1. **Sampling** – the process of converting a continuous time signal to discrete time signal so that computers can process the data digitally.
2. **Aliasing** – the phenomenon where because of too low a sampling frequency, the original signal is corrupted in a way that recovery is not possible.

Continuous time vs Discrete time

- Continuous time system
  - Good for analogue & general understanding
  - Appropriate mostly to analogue electronic systems



- Electronics are increasingly digital
  - E.g. mobile phones are all digital, TV broadcast is will be 100% digital in UK
  - We use digital ASIC chips, FPGAs and microprocessors to implement systems and to process signals
  - Signals are converted to numbers, processed, and converted back

We have looked at ADC, DAC, and quantization in Lecture 2, and in the Electronics 1 module last year. The essence of this slide is to remind you that we sample the data (discrete time) and quantise the samples (discrete voltage steps) using a ADC. We turn digital samples back to analogue voltages using a DAC.

Note that the notation for continuous time and discrete time is different:

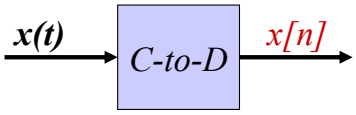
continue time -> discrete time

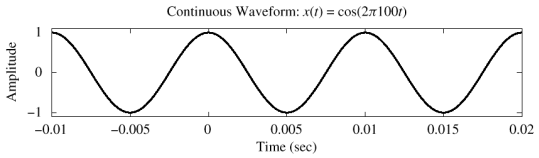
x(t) ➔ x[n]

Sampling Process

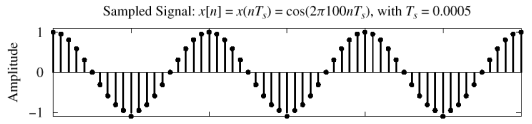
- Use A-to-D converter to turn x(t) into numbers x[n]
- Take a sample every sampling period T_s – uniform sampling

$x[n] = x(nT_s)$

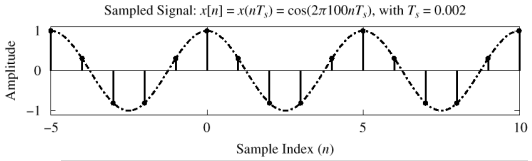




signal frequency $f = 100\text{Hz}$



sampling frequency $f_s = 2\text{kHz}$



sampling frequency $f_s = 500\text{Hz}$

Again you should be familiar with the sampling process already. The important point to note here is that we currently only consider UNIFORM SAMPLING, meaning that we take a sample of the signal x(t) at **regular intervals** of period Ts, where 1/Ts = sampling frequency fs.

It is possible to perform non-uniform sampling. However, the maths for that is really complicated. Therefore, practical electronic systems always use uniform sampling.

The next important point is to ask the question: HOW OFTEN DO WE NEED TO TAKE SAMPLES? Obviously if we take a very large number of samples with very small Ts is clearly wasteful. Sampling very infrequently is also obviously bad – we will miss important features in the signal, and therefore lose information.

Remember from an earlier lecture and from last year, you have been told an important principle:

1. **Sampling**, done properly, will NOT destroy information. That is, you can always get back the original signal with no loss of information.
2. **Quantization** will almost always destroy information. It introduces errors, known as **quantitation noise**, that cannot be avoided.

The first statement is only true if sampling is done properly in the sense that you have retained enough samples of the original signals. "How much is enough?" is therefore an important question that requires a precise and definitive answer.

Sampling Theorem

- Bridge between continuous-time and discrete-time
- Tell us HOW OFTEN WE MUST SAMPLE in order not to loose any information

Sampling Theorem

A continuous-time signal x(t) with frequencies no higher than f_{max} (Hz) can be reconstructed EXACTLY from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{max}$.

- For example, the sinewave on previous slide is 100 Hz. We need to sample this at higher than 200 Hz (i.e. 200 samples per second) in order NOT to loose any data, i.e. to be able to **reconstruct** the 100 Hz sinewave exactly.
- fmax refers to the maximum frequency component in the signal that has **significant** energy.
- Consequence of violating sampling theorem is **corruption of the signal** in digital form.

Sampling Theorem (sometimes also known as the Shannon Theorem or the Nyquist Theorem) provides the answer. It states that if the original signal has a MAXIMUM frequency component at f_{max}, then we MUST sample at **2 x f_{max} or higher** in order to avoid corrupting the signal.

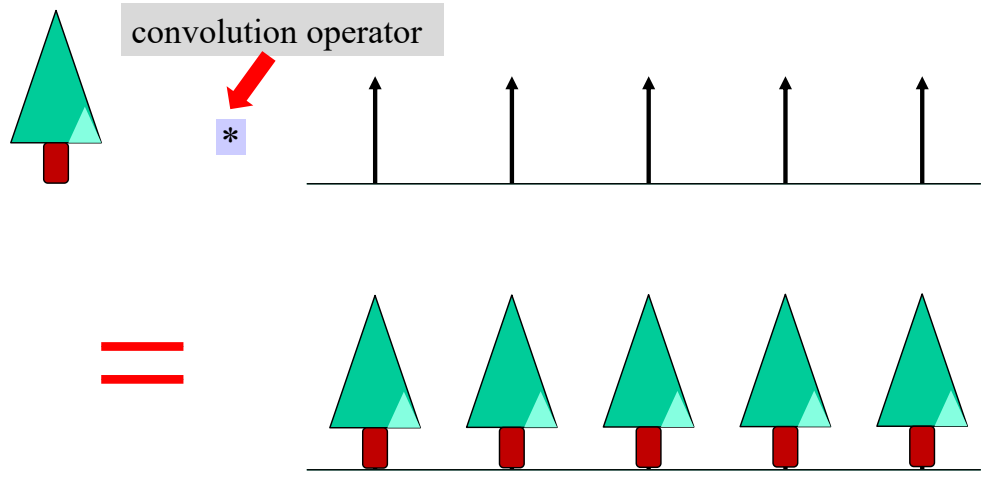
Simply put,

Sampling frequency fs ≥ 2 fmax

What if you ignore this rule and sample the signal too infrequently? You will corrupt the signal by introducing spurious signals through a phenomenon known as "**aliasing**". We will consider aliasing later and in Lab 2.

Intuitive idea of convolution

- Convolution – an important concept in signal processing
- Example: planting tree in a row, at regular interval



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Before you can understand WHY sampling theorem is true, you need to understand an linear operator known as **convolution**.

Convolution is a topic that I will explain in some details later in this module. In the meantime, you only need an intuitive understanding of what this operation is and how it is applied.

Convolution is a special form of multiplication and integration. An example is shown in the slide. You want to plan identical tree (it is sometimes called a **kernel**) along a row in a garden at regular interval. The side view of the tree is as shown here. The location where planting is to take place is shown as sequence of impulses.

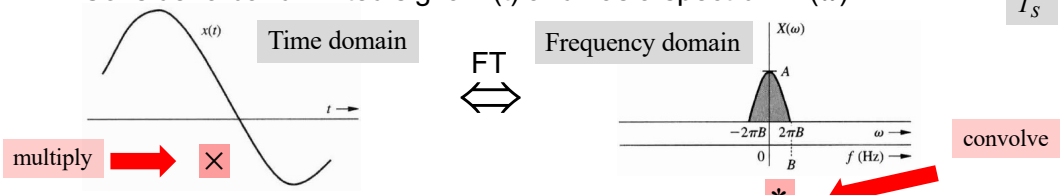
The question is: what is the side view of the garden after you have planted these five trees?

Of course, the result is as shown in the lower picture. The operation between the kernel (tree) and the planting location (sequence of 5 impules) is similar (but not exactly the same) as the operation of convolution.

Now we are ready to explain what sampling does to a signals frequency spectrum and why Sample Theorem is true.

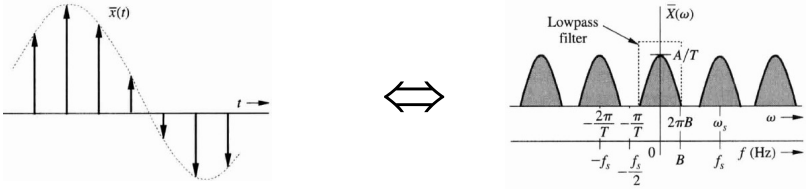
Sampling Theorem: Intuitive proof (1)

- Consider a band limited signal x(t) and has a spectrum X(ω):



- Ideal sampling = multiply x(t) with impulse train :

- Therefore the sampled signal has a spectrum (convolution):



We can mathematically prove what happens to a signal when we sample it in both the time domain and the frequency domain, hence derive the Sampling Theorem.

Instead of doing this in maths, I will use only what we have covered in this module so far, and demonstrate Sampling Theorem through deduction with pictures only.

Let us consider an original continuous time (CT) signal x(t). Assume that its energy is limited to frequency of ±B Hz.

If we sample x(t) at a frequency fs, where fs = 1/Ts and Ts is the sampling period, we get the bottom left waveform, which is the result fo multiplying x(t) with the sampling impulse train as shown here. (Lecture 4, idea 3.). Note that the spectrum of the impulse train is also an impulse train in frequency domain. Further, Ts is inverse proportional to ω_s.

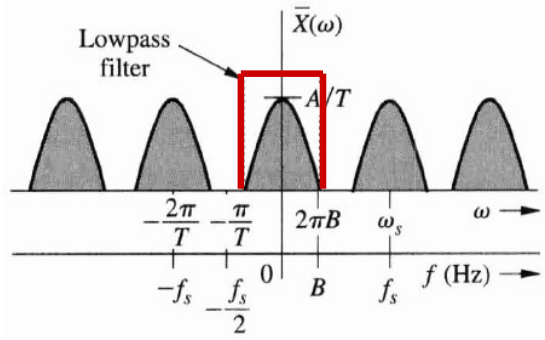
It will be shown in later lectures one very important principle:

MULTIPLYING TWO SIGNALS IN THE TIME DOMAIN IS EQUIVALENT TO CONVOLUTING THEIR SPECTRA IN THE FREQUENCY DOMAIN

The effect of **convoluting** the spectrum of the sampling impulse train with the original signal spectrum X(ω) is to **replicate** the shape of $X(\omega)$ wherever there is an impulse in the frequency spectrum as shown in the bottom right figure. Why this is the case is not important at the moment. For now, just accept this as a fact that follows from the statement above.

Sampling Theorem: Intuitive proof (2)

- Therefore, to reconstruct the original signal x(t), we can use an ideal lowpass filter on the sampled spectrum:



- This is only possible if the shaded parts do not overlap. This means that f_S must be more than TWICE that of B.

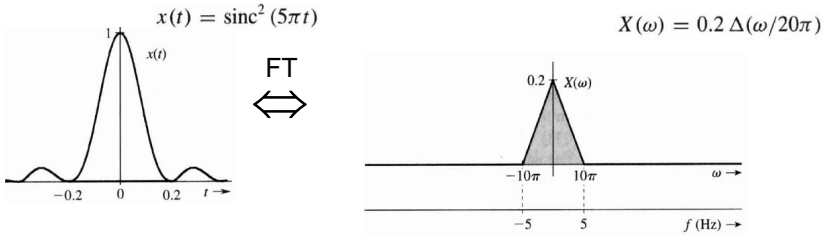
Note that from this spectral diagram, you can see that the original signal x(t) is represented by the main shape in the spectral domain around frequency ω = 0. Sampling add all the other "***spurious***" parts to the spectrum, which were not found in the original spectrum, the result of **convoluting** $X(\omega)$ with the **impulse train** in the frequency domain to give us the new spectrum of the sampled signal.

There are two implications:

1. The first is that, if we apply a filter to the sampled signal, and extract the main spectral components within the RED rectangle in the frequency domain, we can recover the original signal with no loss of information. Here we show that the filtering process is a lowpass filter, where we only retain frequency components within ±B Hz, and remove all other high frequency components. This is known as "**reconstruction**".
2. In order to have perfect reconstruction, the neighbouring spectral contents MUST NOT overlap with the main spectrum. This is only guaranteed if fs ≥ 2B – which is the sampling theorem itself! (That is, we must sample the signal at a rate higher than **twice** that of the signal bandwidth.)

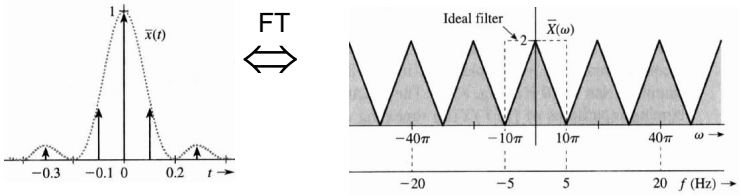
What happens if we sample too slowly? (1)

- What are the effects of sampling a signal at, above, and below the Nyquist rate? Consider a signal bandlimited to 5Hz:



- Sampling at Nyquist rate of 10Hz give:

perfect reconstruction possible



Let us take another look at what happens if we sample too infrequently. Let us consider a concrete example of a band limited signal, where all signal energy is within 5Hz. The spectrum of the signal x(t) is as show here – a triangular spectrum between ±5Hz.

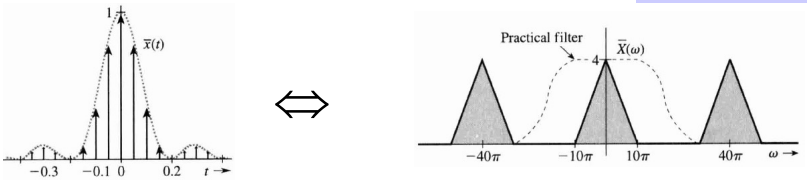
If we sample the signal at 10Hz, we obtain a series of impulses on the bottom left. Using what we know so far, we can predict that the spectrum of the sampled version of the original signal is as shown here.

Now we can get back the original spectrum if we apply a perfect reconstruction filter that eliminates ALL frequency components outside ±B Hz as shown.

What happens if we sample too slowly? (2)

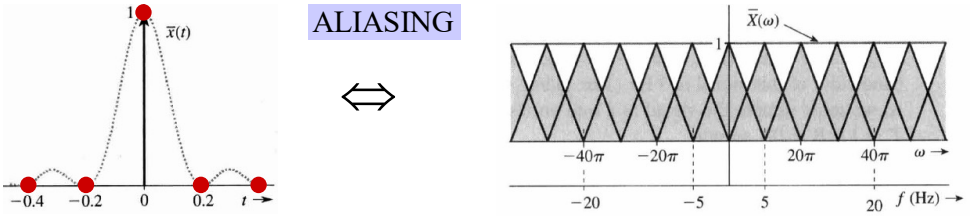
- Sampling at higher than Nyquist rate at 20Hz makes reconstruction much easier.

perfect reconstruction practical



- Sampling below Nyquist rate at 5Hz corrupts the signal.

ALIASING



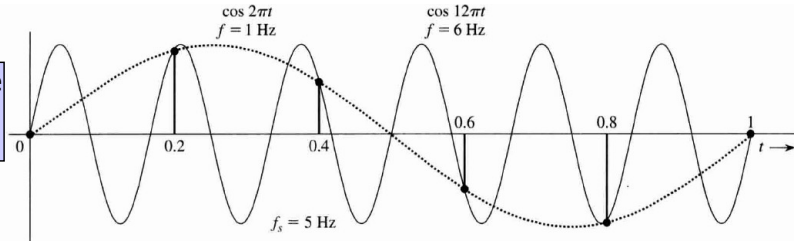
If we sample at higher than 10 Hz, then the effect is to push apart the triangles as shown, with gaps between them in the frequency spectrum. This makes reconstruction easier – we can use a lowpass filter that is easy and economical to implement as shown. This is what we NORMALLY do. We usually sample the signal with, say fs ≥ 4 x fmax, as shown here in order to make reconstruction easy.

However, if we sample the original signal at 5Hz,we will miss all the details and only see an impulse at t=0. This is because the spectral shapes now overlap each other AND corrupt the signal in a way that there is NO WAY BACK! Indeed, we see the the result is a spectrum that is constant at all frequency – the spectrum of an impulse at t= 0!

Spectral folding effect of Aliasing

- Consider what happens when a 1Hz and a 6Hz sinewave is sampled at a rate of 5Hz.

1Hz & 6Hz sinewaves are indistinguishable after sampling



- In general, if a sinusoid of frequency f Hz is sampled at fs samples/sec, then sampled version would appear as samples of a continuous-time sinusoid of frequency $|f_a|$ in the band 0 to fs/2, where:

$$|f_a| = |f \pm mf_s| \quad \text{where} \quad |f_a| \le \frac{f_s}{2}, \ \text{m is an integer}$$

- In other words, the 6Hz sinusoid is FOLDED to 1Hz after being sampled at 5Hz.

This phenomenon is known as **aliasing** – which is a fancy term to describe the spectral folding effect of under-sampling.

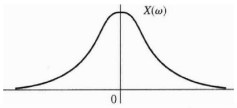
Consider two sinusoidal signals, one at 1Hz and another at 6Hz. If the signal is sampled at a rate of 5Hz, we will GET EXACTLY THE SAME sample values for both signals. In other words, we CANNOT tell whether the original signal is at 1Hz or is at 6Hz. After sampling, they both yield the same sampled values.

In other words, if we sample a sinusoid at frequency fa using sampling frequency of fs, any frequencies components f = fa ± mfs will appear as fa after sampling. So 6Hz, 11Hz, 16Hz …. will all appears to be a 1Hz sinusoid after sampling – we cannot tell them apart!

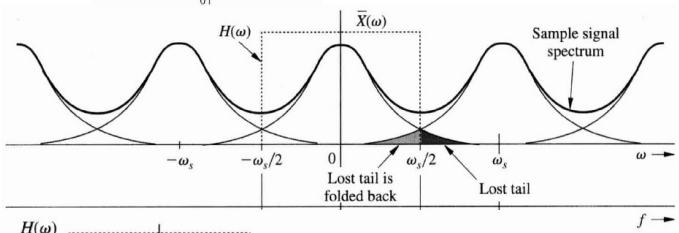
This **spectral folding** is known as **aliaising**.

Anti-aliasing filter (1)

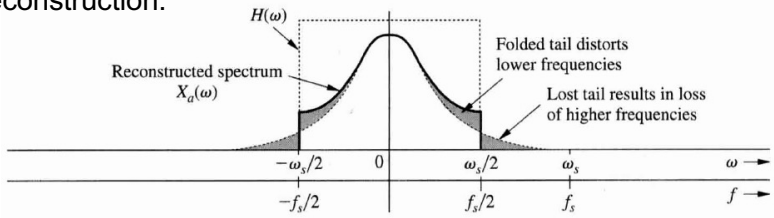
- To avoid corruption of signal after sampling, one must ensure that the signal being sampled at fs is bandlimited to a frequency B, where B < fs/2.
- Consider this signal spectrum:



- After sampling:



- After reconstruction:

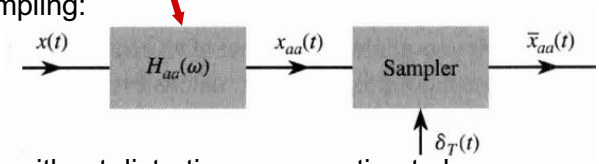


The consequence of this theorem is that, in order to avoid corruption of the information (signal), we MUST limit the bandwidth of the signal, BEFORE sampling taking place. In the diagram here, the original signal has very small energy at the tail of the spectrum as shown.

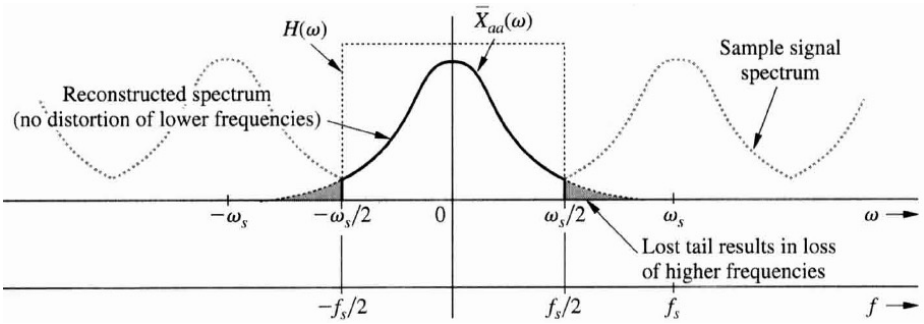
After sampling, the tail energy components (shaded part) get folder back to the main spectral region, and corrupts part of the signal.

Anti-aliasing filter (2)

- Apply a lowpass filter before sampling:



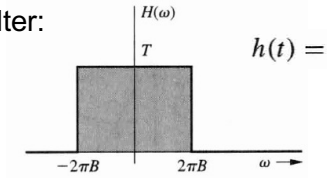
- Now reconstruction can be done without distortion or corruption to lower frequencies:



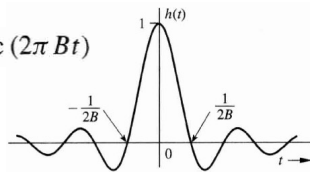
In order to avoid such corrupt, we must always apply a lowpass filter to the continuous time signal BEFORE sampling is performed. This cuts off the energy beyond fs/2, and therefore avoid corrupting the original signal spectrum (and hence the original signal) because now there is NO aliasing or spectral folding.

Ideal Signal Reconstruction

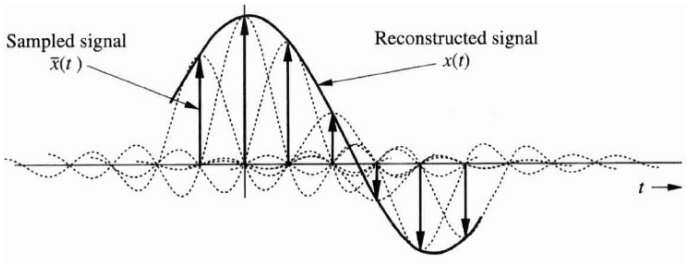
- Use ideal lowpass filter:



$$h(t) = \text{sinc}\,(2\pi Bt)$$



- That's why the sinc function is also known as the **interpolation** function:



$$x(t) = \sum_n x(nT)h(t - nT)$$

$$= \sum_n x(nT)\,\text{sinc}\,(2\pi Bt - n\pi)$$

We have seen that we can get back the original continuous time signal x(t) from the sampled version by applying an ideal lowpass filter which as rectangular spectrum as shown. That is to say, we MULTIPLY the sampled signal spectrum with the rectangular filter, and the result is the spectrum of the original signal X(ω).

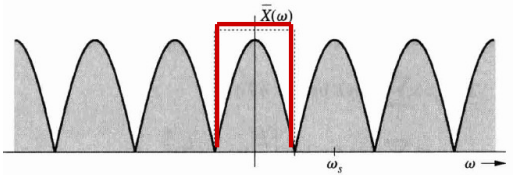
The time-domain version of the rectangular filter is a sinc function! Because of the spectral shape has abrupt edges (fast changes) in frequency, it results in the time domain equivalent having signals at large value of t (i.e. goes on forever).

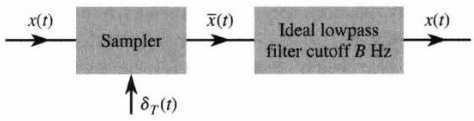
Here is another general principle: abrupt changes in time cause high frequency components in frequency domain; converse abrupt changes in frequency domain cause time domain signals that goes on for a long time (i.e. high time values).

The sinc function is also known as the interpolation function – we could use this to reconstruct (interpolate) the samples and get back the original signal. However, sinc function goes on forever – no very practical to use in real systems.

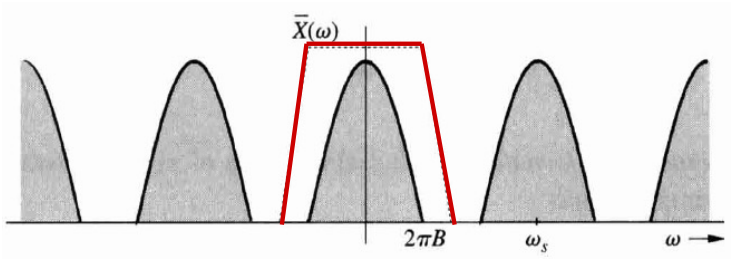
Practical Signal Reconstruction

- Ideal reconstruction system is therefore:





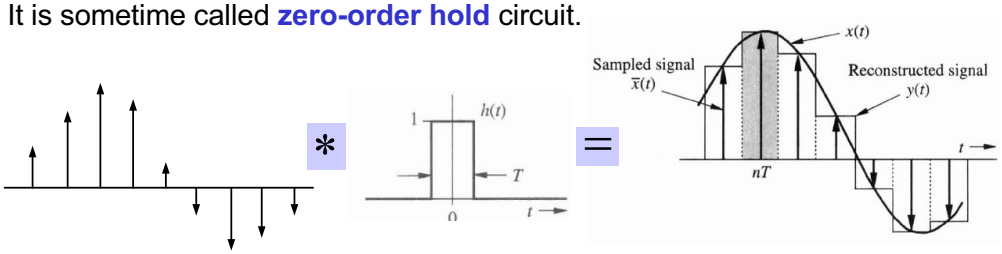
- In practice, we normally sample at higher frequency than twice f_{max}:



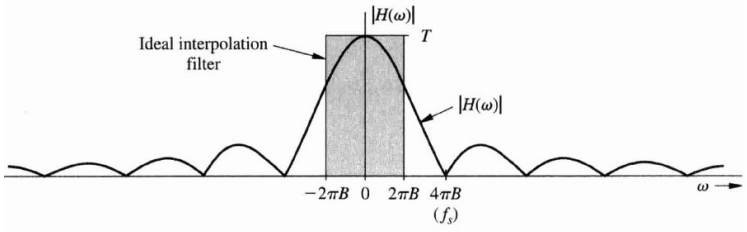
So in practice, we DON'T use the sinc function for reconstruction. Instead, we sample the signal at higher than twice fmax (i.e. fs >> 2 fmax), introducing the gaps, thus allowing us to use filter with less abrupt cutoff characteristics in the frequency domain.

Signal Reconstruction using D/A converter

- D/A converter is a simple interpolator that performs the job of signal reconstruction.
- It is sometime called **zero-order hold** circuit.



- The effect of using the D/A converter is a non-ideal lowpass filter.



In real electronics, we use a DAC to perform the reconstruction of the continuous time signal from discrete signal samples. Without proving anything, what this does is to "convolve" the discrete signal samples with a pulse signal in the time domain. This is equivalent to multiplying (apply) a filter with a sinc function in the frequency domain as shown here. While this is NOT ideal (as shown in the rectangular filter), it is close. Further, it will leak some of the high frequency components (hence the output of a DAC is rugged). We can remove these by following the DAC with a simple RC type lowpass filter to get rid of those high frequencies.

Three Big Ideas

1. **Sampling Theorem** tells us that we MUST sample a signal at a frequency that is higher than **TWICE the maximum signal frequency** to avoid corruption of the signal.

2. **Multiplication** in the **time domain** is the same as **convolution** in the **frequency domain**.

3. **Sampling changes the frequency spectrum** of the original signal – it introduces duplicate spectra of the original at 2fs, 3fs ……

Here are the three main ideas in this lecture that is worth you understanding and remembering.